Locally definable fiber bundles

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Abstract

Let $\eta = (E, p, Y, F, K)$ be a locally definable fiber bundle and $f, h : X \to Y$ two locally definable maps. If f and h are locally definably homotopic, then $f^*(\eta)$ and $h^*(\eta)$ are locally definably fiber bundle isomorphic.

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1 . Introduction.

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers. General references on o-minimal structures are [2], [5], see also [15]. For example, the Nash category is a special case of the definable C^r category and it coincides with the definable C^∞ category based on $\mathcal{R} = (\mathbb{R}, +, \cdot, >)$ ([17]). J.P. Rolin, P. Speissegger and A.J. Wilkie [14] proved that there exist uncountably many o-minimal expansions of $\mathcal{R}$. Further properties and constructions of them are studied in [3], [4], [6], [13]. Equivariant definable category is studied in [8], [10], [11].

In this paper "definable" means "definable with parameters in $\mathcal{M}$", everything is considered in $\mathcal{M}$, "countable" means finite or countably infinite and each locally definable map is continuous unless otherwise stated.

A subset X of $\mathbb{R}^n$ is called *locally definable* if for every $x \in X$ there exists a definable open neighborhood U of x in $\mathbb{R}^n$ such that $X \cap U$ is a definable subset of X. A more general setting of locally definable sets is studied in [1]. Clearly every definable set is locally definable, every compact locally definable set is definable and any open subset of $\mathbb{R}^n$ is locally definable.

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be locally definable sets. We say that a continuous map $f : U \to V$ is a *locally definable map* if for any $x \in U$ there exists a definable open neighborhood W_x of x in $\mathbb{R}^n$ such that $f|U \cap W_x$ is definable.

For example, if $\mathcal{M} = \mathbf{R}_{an}$, then $f : (-1, 1) \to \mathbb{R}, f(x) = \sin \frac{1}{1-x^2}$ is locally definable but not definable.

As a generalization of definable fiber bundles introduced in [10], we can define *locally definable fiber bundles* (See Definition 2.1). Two locally definable maps $f, h : X \to Y$ between locally definable sets are *locally definably homotopic* if there exists a locally definable map $H : X \times [0, 1] \to Y$ such that

$H(x, 0) = f(x)$ for all $x \in X$ and $H(x, 1) = h(x)$ for all $x \in X$.

Theorem 1.1. *Let $\eta = (E, p, Y, F, K)$ be a locally definable fiber bundle and $f, h : X \to Y$ two locally definable maps. If f is locally definably homotopic to h, then $f^*(\eta)$ and $h^*(\eta)$ are locally definably fiber bundle isomorphic.*

Theorem 1.1 is a locally definable version of 1.2 [7].

The following result is a classification theorem of principal fiber bundles (19.3 [16]).

Theorem 1.2 (19.3 [16])**.** *Let K be a compact Lie group, γ_{n+1} the $(n+1)$-universal principal K fiber bundle whose base space is Y. Let X be an n-complex. Then the operation of assigning to each continuous map $f : X \to Y$ its induced fiber bundle sets up to a bijective correspondence between homotopy classes of continuous maps of X to Y and fiber bundle isomorphism classes of principal bundles over Y with structure group K.*

If K is a compact definable group, then by the construction of an $(n + 1)$-universal principal K fiber bundle (19.6 [16]), it is a definable principal K fiber bundle.

The following is a definable version of Theorem 1.2.

Theorem 1.3. *Let K be a compact definable group, γ_{n+1} the $(n+1)$-universal principal K fiber bundle whose base space is Y. Let X be an n-dimensional definable set. Then the set of homotopy classes of definable maps between X and Y corresponds bijectively to the set of definable fiber bundle isomorphism classes of principal K definable fiber bundles over X.*

2 Proof of results

Remark that for any locally definable map f between locally definable sets X and Y, if X is compact, then $f(X)$ is a definable set and $f : X \to f(X)$ ($\subset Y$) is a definable map.

Note that the maps $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = \sin x, f_2(x) = \cos x$, respectively, are analytic but not locally definable in $\mathcal{R} = (\mathbb{R}, +, \cdot, >)$, and that the field $\mathbb{Q}$ ($\subset \mathbb{R}$) of rational numbers is not a locally definable subset of $\mathbb{R}$.

A topological space X is a *locally definable space* if there exists a countable family of charts $\{(U_i, \phi_i)\}_{i \in I}$, where U_i is an open subset of X and ϕ is homeomorphism from U_i to a definable subset Z_i of $\mathbb{R}^n$ for all i, such that $\{U_i\}$ is an open covering of X and for each pair $(i, j) \in I \times I$ $\phi_i(U_i \cap U_j)$ is a definable open subset of Z_i and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable homeomorphism.

Definition 2.1 ([10])**.** (1) A topological fiber bundle $\eta = (E, p, X, F, K)$ is called a *locally definable fiber bundle* over X with fiber F and structure group K if the following two conditions are satisfied:

(a) The total space E is a locally definable space, the base space X is a locally definable set, the structure group K is a definable group, the fiber F is a locally definable set with an effective locally definable K action, and the projection $p : E \to X$ is a locally definable map.

(b) There exists a countable family of local trivializations $\{U_i, \phi_i : p^{-1}(U_i) \to U_i \times F\}_i$ of η such that each U_i is a definable open subset of X, $\{U_i\}_i$ is a countable open covering of X. For any $x \in U_i$, let $\phi_{i,x} : p^{-1}(x) \to F, \phi_{i,x}(z) = \pi_i \circ \phi_i(z)$, where π_i stands for the projection $U_i \times F \to F$. For any i and j with $U_i \cap U_j \neq \emptyset$, the transition function $\theta_{ij} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \to K$ is a definable map. We call these trivializations *locally definable*.

Locally definable fiber bundles with compatible definable local trivializations are identified.

(2) Let $\eta = (E, p, X, F, K)$ and $\zeta = (E', p', X', F, K)$ be locally definable fiber bundles whose locally definable local trivializations are $\{U_i, \phi_i\}_i$ and $\{V_j, \psi_j\}_j$, respectively. A locally definable map $\overline{f} : E \to E'$ is said to be a *locally definable fiber bundle morphism* if the following two conditions are satisfied:

 (a) There exists a locally definable map $f : X \to X'$ such that $f \circ p = p' \circ \overline{f}$.

 (b) For any i, j such that $U_i \cap f^{-1}(V_j) \neq \emptyset$ and for any $x \in U_i \cap f^{-1}(V_j)$, the map $f_{ij}(x) := \psi_{j,f(x)} \circ \overline{f} \circ \phi_{i,x}^{-1} : F \to F$ lies in K, and $f_{ij} : U_i \cap f^{-1}(V_j) \to K$ is a definable map.

A bijective locally definable fiber bundle morphism $\overline{f} : E \to E'$ is called a *locally definable fiber bundle equivalence* if f is a locally definable homeomorphism and $(\overline{f})^{-1} : E' \to E$ is a locally definable fiber bundle morphism covering $f^{-1} : X' \to X$. A locally definable fiber bundle equivalence is a *locally definable fiber bundle isomorphism* if $X = X'$ and $f = id_X$. We say that η is *locally definably trivial* if η is locally definably fiber bundle isomorphic to the trivial bundle $(X \times F, proj, X, F, K)$, where $proj : X \times F \to X$ denotes the projection onto the first factor.

(3) A continuous section $s : X \to E$ of a locally definable fiber bundle $\eta = (E, p, X, F, K)$ is a *locally definable section* if for any i, the map $\phi_i \circ s|U_i : U_i \to U_i \times F$ is a definable map.

(4) We say that a locally definable fiber bundle $\eta = (E, p, X, F, K)$ is a *principal locally definable fiber bundle* if $F = K$ and the K action on F is defined by the multiplication of K.

Theorem 2.2. *Let $\eta = (E, p, X, F, K), \eta' = (E', p', X', F, K)$ be two locally definable fiber bundles. Let $\overline{h_0} : E \to E'$ be a locally definable fiber bundle morphism and $H : X \times [0, 1] \to X'$ a locally definable homotopy of the induced map $h_0 : X \to X'$. Then there exists a locally definable homotopy $\overline{H} : \eta \times [0, 1] \to \eta'$ of $\overline{h_0}$ whose induced homotopy is H.*

Since a locally definable set X is paracompact, for any countable definable open cover $\{U_\alpha\}$ of X, there exists a partition of unity $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$ such that each f_α is locally definable. Thus we have the following lemma.

Lemma 2.3. *Let X be a locally definable set and V_1, V_2 two locally definable open subsets of X with $\overline{V_1} \subset V_2$, where $\overline{V_1}$ denotes the closure of V_1 in X. Then there exists a locally definable function $\rho : X \to [0, 1]$ such that $\rho(\overline{V_1}) = 1$ and $\rho(X - V_2) = 0$. Moreover if X is definable and V_1, V_2 are definable, then we can take ρ to be definable.*

Proof of Theorem 2.2. We first prove the case where X is compact. Let $\{U_\lambda, \psi_\lambda\}$, $\{V_j, \phi_j\}$ be a finite family of definable trivializations of η, η', respectively. Since X is compact, $X \times [0, 1]$ is compact. Thus there exists a refinement $\{U_\lambda \times I_\nu\}$ of a definable open cover $\{H^{-1}(V_j)\}$ such that $\{I_\nu\}_{\nu=1}^r$ is an open cover of $[0, 1]$ and that for each ν with $2 \leq \nu \leq r - 1$ I_ν only intersects $I_{\nu-1}$, $I_{\nu+1}$. We take real numbers $t_0, \ldots, t_r$ such that $0 = t_0 < t_1 < \cdots < t_r = 1$ and $t_\nu \in I_\nu \cap I_{\nu+1}$.

We now proceed by induction on ν. Assume that we have defined $\overline{H_\nu} : E \times [0, t_\nu]$. For any $x \in X$, we can take definable open sets W, W' such that $x \in W \subset \overline{W} \subset W' \subset \overline{W'} \subset U_\lambda$ for some λ, where $\overline{W}$ denotes the closure of W in X. Since X is compact, there exist a finite cover $\{W_\alpha\}_{\alpha=1}^s$ of X. By Lemma 2.3, there exists a definable map $u_\alpha : X \to [t_\nu, t_{\nu+1}]$ such that $u_\alpha(\overline{W_\alpha}) = t_\nu, u_\alpha(X - W'_\alpha) = t_{\nu+1}$. Define $\tau_0 = t_\nu, \tau_\alpha = \max(u_1(x), \ldots, u_\alpha(x))$. Then $t_\nu = \tau_0(x) \leq \cdots \leq \tau_s(x) = t_{\nu+1}$. Let $X_\alpha = \{(x, t) \in X \times [0, 1] | t_\nu \leq t_{\nu+1}\}$. Then $X \times \{t_\nu\} = X_0 \subset \cdots \subset X_s = X \times [t_\nu, t_{\nu+1}]$. Let $E_\alpha = q^{-1}(X_\alpha)$, where $q : E \times [0,1] \to X \times [0,1], q(x,t) = (p(x),t)$. Then $E_\alpha \times \{t_\nu\} = E_0 \subset \cdots \subset E_s = E \times [t_\nu, t_{\nu+1}]$. Let $E_\alpha = \eta \times [0,1] | q^{-1}(X_\alpha)$. Assume that we have a required map $\overline{H_{\alpha-1}}$ on $E_{\alpha-1}$. We now extend $\overline{H_{\alpha-1}}$ to E_α. By the definition of τ and W, $X_\alpha - X_{\alpha-1} \subset W'_\alpha \times [t_\nu, t_{\nu+1}], \overline{W'_\alpha} \times [t_\nu, t_{\nu+1}] \subset U_\alpha \times I_\nu, H(\overline{W'_\alpha} \times [t_\nu, t_{\nu+1}] \subset V_j$. Define $\overline{H_\alpha}(e,t) = (\phi_j(H(x,t), p_j(\overline{H_{\alpha-1}}(e, \tau_{\alpha-1}(x))))), x = q(e), (x,t) \in X_\alpha - X_{\alpha-1}$. Then $\overline{H_\alpha}$ is definable, $p' \circ \overline{H_\alpha}(e,t) = h(x,t)$ and if $t = \tau_{\alpha-1}$ then it coincides with $\overline{H_{\alpha-1}}(e, \tau_{\alpha-1}(x))$. Moreover $\overline{H_\alpha}$ is a definable fiber bundle morphism. Thus in this case the result is proved.

We now prove the general case. Since X is locally definable, there exist a countable family of definable open sets $\{W_n\}_{n=1}^\infty$ such that $\overline{W_n} \subset W_{n+1}$, each $\overline{W_n}$ is compact and $X = \cup_{n=1}^\infty W_n$. By Lemma 2.3, there exists a locally definable map $\tau_n : X \to [0,1]$ such that $\tau_n(\overline{W_n}) = 1$ and $\tau_n(X - W_{n+1}) = 0$. Let $\tau_0 = 0$. Then for each n $\tau_n(x) \le \tau_{n+1}(x)$ and for any $x \in X$, there exists an n with $\tau_n(x) = 1$. Let $X_n = \{(x,t) \in X \times [0,1] | 0 \le t \le \tau_n(x)\}$ and $\eta_n = (E_n, p_n, X_n, F, K)$ the restriction of $\eta \times [0,1]$ on X_n. Then $\overline{h_0} : E \to E'$ is a locally definable fiber bundle morphism. Let $\overline{H_{n-1}} : \eta_{n-1} \to \eta$ be a locally definable fiber bundle morphism expanding $\overline{h_0}$ and $A = \overline{W_{n+1}} - W_{n-1}$. Then A is compact and $x \in A$ if $\tau_{n-1}(x) < \tau_n(x)$. We define $\lambda_x : [0,1] \to [\tau_{n-1}(x), \tau_n(x)]$ by $\lambda_x(s) = s\tau_n(x) + (1-s)\tau_{n-1}(x)$. Then the inverse of λ_x is defined by $\lambda_x(t) = \frac{t - \tau_{n-1}(x)}{\tau_{n-1}(x) - \tau_{n-1}(x)}$. For $x \in A, e \in q^{-1}(A)$, we define $\overline{H_0}(e) = \overline{H_0}(e, \tau_{n-1}(q(e)), h'(x,s) = h(x, \lambda_x(s))$.
Then $\overline{H'_0}$ is a locally definable fiber bundle morphism and h' is a definable homotopy of H_0. Applying the compact case, we have a definable homotopy $\overline{H'}$. Define $\overline{H'_n}(e,t) = \overline{H'}(e, \lambda_x^{-1}(t)), x = q(e), \tau_{n-1}(x) < t \le \tau_n(x)$. Combining $\overline{H_{n-1}}$, we have a locally definable homotopy $\overline{H_n}$ extending $\overline{H_{n-1}}$. By induction, we have the required locally definable homotopy $\overline{H}$. □

Theorem 1.1 follows from Theorem 2.2. □

Theorem 2.4 (8.2.9 [2]). *(Definable triangulation theorem) Let X be a definable set and $X_1, \ldots, X_k$ definable subsets of X. Then there exists a definable triangulation (M, τ) of X compatible with $X_1, \ldots, X_k$, namely M is a simplicial complex and τ is a definable homeomorphism from X to a union of open simplexes of M such that each $\tau(X_i)$ is a union of open simplexes of M. In particular, if X is compact, then $\tau(X) = |M|$.*

Theorem 2.5 (1.2 [8]). *Let X, Y be definable sets. Then the set of homotopy classes of continuous maps between X and Y corresponds bijectively to the set of definable homotopy classes of definable maps between X and Y.*

Proof of Theorem 1.3. By Theorem 2.4, X admits a definable triangulation. Thus X is an n-complex.

Let η be a principal K definable fiber bundle over X. By Theorem 1.2, there exists a continuous map $f : X \to Y$ such that η is fiber bundle isomorphic to $f^*(\gamma_{n+1})$. By Theorem 2.5, we have a definable map $h : X \to Y$ which is homotopic to f. By [12], $f^*(\gamma_{n+1})$ is fiber bundle isomorphic to $h^*(\gamma_{n+1})$. Thus η and $h^*(\gamma_{n+1})$ are fiber bundle isomorphic. By 1.2 [9], they are definably fiber bundle isomorphic. Moreover by Theorem 2.5, a definable homotopy representative of f is unique up to definable homotopy. □

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